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ALGEBRAIC SYSTEMS FOR DIGITAL SIGNAL PROCESSING

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Abstract. A digital signal processing based on a representation over various algebraic systems is discussed. Theorems of spectral decomposition of a multiple-valued function are formulated. Examples of the function decomposition are given.

1. INTRODUCTION

Image analysis, signal processing, logic design are normally thought of in terms of multiple-valued signals; however it is natural to think of variables with symbolic or integer values. For that a multiple-valued signal f is transformed into the spectral representation by the discrete transformation

$$f(t)=\sum_{i=0}^{m-1}\theta(t,i)\cdot a(i),\; a(i)=\sum_{t=0}^{m-1}\vartheta(i,t)\cdot f(t),\; \sum_{i=0}^{m-1}\vartheta(i,\tau)\cdot\theta(t,i)=\begin{cases}1, t=\tau\\ 0, t\neq\tau\end{cases}, \quad (1)$$

where $f(t)$ are digital readouts of a signal in sampling instants $t=\overline{0,m-1}$; $a(i)$ ($i=\overline{0,m-1}$) is a spectrum of the signal; $\theta(t,i)$ and $\vartheta(i,t)$ are a system of orthogonal signals (functions); m is a number of readouts in the temporal and the spectral (frequency) area.

The spectral transformation (1) based on the operations of addition and multiplication. For binary functions are used conjunction and disjunction [1], conjunction and nonequivalence [2] and arithmetic operations [3]. For multiple-valued functions are used maximum and minimum [4], addition modulo k and minimum [5], arithmetic addition and digit-to-digit operations (conjunction, disjunction and nonequivalence) [6], operations of ring of integers [7] and operations of finite fields [8].

In this paper algebraic systems for spectral decomposition are generalized.

2. NOTATION

Let a domain N_k is a finite set of integers $\{0,1,\ldots,k-1\}$ and a multiple-valued variable x_i can take on values from N_{k_i}. A discrete function or k_f-valued m-function f is a function, which maps domain N_m to domain N_{k_f}, formally, $f:N_m\to N_{k_f}$. If $m=k_0k_1\ldots k_{n-1}$ then any m-function f can be represented as $f:N_{k_0}\times N_{k_1}\times\ldots\times N_{k_{n-1}}\to N_{k_f}$. We determine linkage between value i of variable x and values i_j of variables x_j ($j=\overline{0,n-1}$) by the $k_{n-1}\ldots k_1k_0$-ary expansion of i, $i=(i_{n-1},\ldots,i_1,i_0)_{k_{n-1}\ldots k_1k_0}$, where i_0 is the least significant digit.

Example 1. An example of discrete function $F=[3\,0\,1\,2\,2\,1]$ is shown in Figure **1**. Assume that $k_0=2$, $k_1=3$, $k_f=4$, $N_{k_0}=\{0,1\}$ $N_{k_1}=\{0,1,2\}$ and $N_{k_f}=\{0,1,2,3\}$.

x_1	0	0	0	1	1	1
x_0	0	1	2	0	1	2
$f(x_0,x_1)$	3	0	1	2	2	1

Figure 1. A 4-valued 6-finction of 2 variables

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3. ALGEBRAIC SYSTEMS

Let $R = \langle N_k, +, \cdot \rangle$ is an algebraic system, where $+$ $(\cdot)$ is called addition (multiplication). Let any function f be depended on two variables: $x' \in N_{k'}$ and $x'' \in N_{k''}$, such that $k'k''' = m$. We write the function f in the form of a sum-of-product expansion over R,

$$f(x', x'') = \sum_{i=0}^{k'-1} \theta_i(x') \cdot a_i(x''), \tag{2}$$

where $k_f \le k$; $a_i \in N_k$ are coefficients (k-valued k''-functions); $\theta_i \in N_k$ are spectral functions (k-valued k''-functions). Equation (2) can be written for each of x' values:

$$\begin{cases} f(0, x'') & = & \theta_0(0) \cdot a_0(x'') & + \ldots + & \theta_{k'-1}(0) \cdot a_{k'-1}(x''); \\ f(1, x'') & = & \theta_0(1) \cdot a_0(x'') & + \ldots + & \theta_{k'-1}(1) \cdot a_{k'-1}(x''); \\ \hdashline f(k'-1, x'') & = & \theta_0(k'-1) \cdot a_0(x'') & + \ldots + & \theta_{k'-1}(k'-1) \cdot a_{k'-1}(x''). \end{cases} \tag{3}$$

The expressions (3) also can be written as a matrix equation $F = D \times A$ (if there exist Q such that $A = Q \times F$, $Q \times D = I$ we have orthogonal transformation), where F (A) is a $k' \times k''$-matrix, D (Q, I) is a direct (inverse, unit) $k' \times k'$-matrix. There are a few algebraic systems, which allow finding a_i from (3).

3.1. Logic algebra

Let $R_L = \langle N_k, +, \cdot \rangle$ be a logic algebra and there exists $\sigma \in N_k$ and $\iota \in N_k$ ($\iota \ne \sigma$) such that $a + \sigma = a$, $\sigma + a = a$ and $\sigma \cdot a = \sigma$, $\iota \cdot a = \iota$ for all $a \in N_k$. Element σ is called zero and element ι is called unit.

Definition 1. A permutation matrix $P_{k'}$ is a $k' \times k'$-matrix, each of whose rows and columns has only one nonzero element and this element is unit.

Theorem 1. Any function f can be decomposed in the form (2) over R_L if D is a permutation matrix $P_{k'}$. Then $Q = D^T$ and $Q \times D = I$, where T denotes a transposition operation of matrix.

Example 2. Let a addition and a multiplication are operations defined by matrix S and U such that $s_{ij} = i + j$ and $u_{ij} = i \cdot j$,

$$S = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & * & * & * \\ 2 & * & * & * \\ 3 & * & * & * \end{bmatrix}, \; U = \begin{bmatrix} 0 & 0 & 0 & 0 \\ * & * & * & * \\ * & * & * & * \\ 0 & 1 & 2 & 3 \end{bmatrix},$$

where * is an indifference value. Obviously $\sigma = 0$ and $\iota = 3$. Then for the function from Example **1** we have

$$D = \begin{bmatrix} \theta_0(0) & \theta_1(0) & \theta_2(0) \\ \theta_0(1) & \theta_1(1) & \theta_2(1) \\ \theta_0(2) & \theta_1(2) & \theta_2(2) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 3 \\ 3 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}, \; Q = \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 3 \\ 3 & 0 & 0 \end{bmatrix},$$

$$A = Q \times \begin{bmatrix} f(0, x'') \\ f(1, x'') \\ f(2, x'') \end{bmatrix}, \; A = \begin{bmatrix} a_0(x'') \\ a_1(x'') \\ a_2(x'') \end{bmatrix} = \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 3 \\ 3 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 3 & 2 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \\ 3 & 2 \end{bmatrix},$$

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$$f(x', x'') = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}(x') \cdot \begin{bmatrix} 0 \\ 2 \end{bmatrix}(x'') + \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}(x') \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}(x'') + \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}(x') \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix}(x''),$$

where $[y_i](x) = y_x$ is a unary operation defined by the vector $[y_i]$.

3.2. Multiplicative algebra

Let $R_M = \langle N_k, +, \cdot \rangle$ be a multiplicative algebra such as R_L. In addition to R_L, let $G_M = \langle N_k \setminus \{\sigma\}, \cdot \rangle$ is a group. In this case for all $a \in G_M$ there exists an inverse element $a^{-1} \in G_M$ such that $a \cdot a^{-1} = \iota$ and $a^{-1} \cdot a = \iota$.

Definition 2. A monomial matrix $M_{k'}$ is a $k' \times k'$-matrix; each of whose rows and columns has only one nonzero element.

Theorem 2. Any function f can be decomposed in the form (2) over R_M if D is a monomial matrix $M_{k'}$. Then $Q = \tilde{D}^T$ and $Q \times D = I$, where $\tilde{D}$ is a matrix each of whose nonzero elements are replaced by its inverse elements.

Example 3. Let operations of R_M are

$$S = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & * & * & * \\ 2 & * & * & * \\ 3 & * & * & * \end{bmatrix}, \; U = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 1 \\ 0 & 3 & 1 & 2 \\ 0 & 1 & 2 & 3 \end{bmatrix}.$$

The function from Example **1** can be written

$$f(x', x'') = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}(x') \cdot \begin{bmatrix} 0 \\ 2 \end{bmatrix}(x'') + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}(x') \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix}(x'') + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}(x') \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}(x'').$$

3.3. Additive algebra

Let $R_A = \langle N_k, +, \cdot \rangle$ be an additive algebra such as R_L. In addition to R_L, let $G_A = \langle N_k, + \rangle$ is a commutative group.

Definition 3. The cyclic order of element $a \in G_A$ is a minimal whole number $c_a > 0$, such that cyclic sum $c_a \circ a = \underbrace{a + a + \ldots + a}_{c_a} = \sigma$, where σ is an identity element of G_A. Let $0 \circ a = \sigma$ and let $(-\lambda) \circ a = \lambda \circ (-a)$ where λ is an integer.

Definition 4. The cyclic order of group G_A is a minimal order of its elements except σ.

Lemma 3. Equation $\lambda \circ a = b$ has unique solution for all $a, b \in G_A$ if and only if $c < |\lambda|$, where c is a cyclic order of commutative group G_A.

Definition 5. A logical matrix $L_{k'}$ is a $k' \times k'$-matrix; each of whose elements is zero or unit. If we replace the elements of $L_{k'}$ with 0 and 1 respectively, we find matrix $\hat{L}_{k'}$. $\hat{L}_{k'}$ is called a conjugate matrix of $L_{k'}$.

Theorem 4. Any function f can be decomposed in the form (2) over R_A if D is a logical matrix $L_{k'}$ and if the modulo of determinant of conjugate matrix $\hat{L}_{k'}$ less then a cyclic

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order of group G_A. Then $\Delta \circ A = \overline{D}^T \circ F$ where $\overline{D}$ is an algebraic complement $\widehat{L}_{k'}$, Δ is a determinant of $\widehat{L}_{k'}$.

Example 4. Let operations of R_A are

$$S = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix},\ U = \begin{bmatrix} 0 & 0 & 0 & 0 \\ * & * & * & * \\ * & * & * & * \\ 0 & 1 & 2 & 3 \end{bmatrix}.$$

A cyclic order of G_A is equal to 2. The function from Example **1** can be written

$$f(x', x'') = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}(x') \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix}(x'') + \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}(x') \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix}(x'') + \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}(x') \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}(x'').$$

3.4. Finite field

Let $R_F = \langle N_k, +, \cdot \rangle$ be a finite field of characteristic p. It is known the field R_F has p^q elements for some positive integer q, if p is a prime number, i.e. $k = p^q$.

Theorem 5. Any function f can be decomposed in the form (2) over R_F if matrix D consist of elements from N_k and if a determinant of D over R_F is not equal to σ. Then $Q = D^{-1}$ and $Q \times D = I$, where D^{-1} is an inverse matrix of D calculated over R_F.

Example 5. Let a field R_F has operations

$$S = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix},\ U = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 1 \\ 0 & 3 & 1 & 2 \\ 0 & 1 & 2 & 3 \end{bmatrix}.$$

The function defined above can be written

$$f(x', x'') = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}(x') \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}(x'') + \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}(x') \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix}(x'') + \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}(x') \cdot \begin{bmatrix} 2 \\ 0 \end{bmatrix}(x'').$$

3.5. Integral domain

A commutative ring $R_I = \langle N_k, +, \cdot \rangle$ with identity is called an integral domain if for all $a, b \in N_k$, $a \cdot b = \sigma$ implies $a = \sigma$ or $b = \sigma$. As is well known any finite integral domain must be a field. Let $R_I = \langle Z, +, \cdot \rangle$ is the ring of integers.

Theorem 6. Any function f can be decomposed in the form (2) over R_I if matrix D consist of elements from Z and if the determinant of D is not equal to zero. Then $\Delta \cdot A = \overline{D}^T \cdot F$, $Q \times D = \Delta \cdot I$ and

$$f(x', x'') = \frac{1}{\Delta} \sum_{i=0}^{k'-1} \theta_i(x') \cdot \dot{a}_i(x''),$$

where $\overline{D}$ is an algebraic complement D, Δ is a determinant of D, $\dot{a}_j = \Delta \cdot a_j$, $\dot{a}_j \in Z$.

Example 6. Our function over the ring of integers can be written

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$$f(x', x'') = -\frac{1}{5} \cdot \left(\begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} (x') \cdot \begin{bmatrix} 2 \\ 10 \end{bmatrix} (x'') + \begin{bmatrix} -2 \\ 0 \\ -1 \end{bmatrix} (x') \cdot \begin{bmatrix} 10 \\ 15 \end{bmatrix} (x'') + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} (x') \cdot \begin{bmatrix} -1 \\ -10 \end{bmatrix} (x'') \right).$$

CONCLUSION

We have five of algebraic systems (logic, multiplicative and additive algebra, finite fields, ring of integer). This systems can be used for syntheses various spectral representations of digital readouts of a signal. As well known difficult problems in time domain can be solved easy in different spectral areas. The given theorems allow constructing spectral functions for digital signal processing.

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