Spectral Methods in Logical Data Processing

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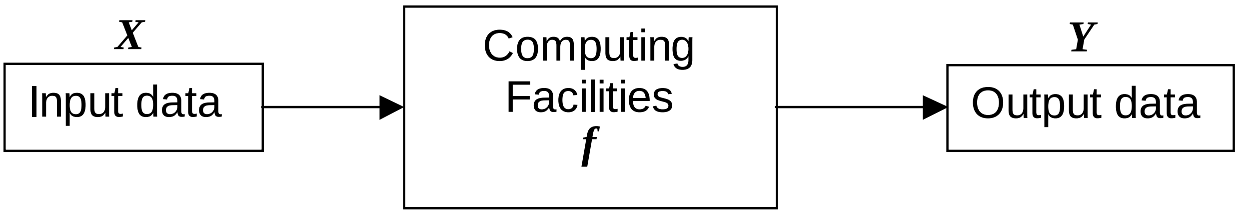
Introduction

Logical data processing methods are widely employed in modern computing devices. They also find extensive application in logical control, digital signal processing, pattern recognition, and in designing discrete devices, etc.



$$f(X) = Y$$

1). The input data X is partitioned into some parts (variables)
2). Variables are transformed into parts of output data by some operations.
3). The parts of output data are formed to output data Y.

$$X \Rightarrow (x_0, x_1, \ldots, x_{n-1}), \qquad \begin{cases} f_0(x_0, x_1, \ldots, x_{n-1}) & = & y_0, \\ f_1(x_0, x_1, \ldots, x_{n-1}), & = & y_1 \\ \ldots & & \ldots \\ f_{t-1}(x_0, x_1, \ldots, x_{n-1}) & = & y_{t-1}; \end{cases} \qquad (y_0, y_1, \ldots, y_{t-1}) \Rightarrow Y.$$

Finally, the problem of logic data processing is a representation of function as a sequence of operations.

Decomposition of functions

$$f(X), \; X = \{x_0, x_1, \ldots, x_{n-1}\}$$

Decomposition	Construction	Sets of variables		
Nonregular	$f(X) = g(\theta(Y), Z)$	$Y \subset X$, $Z \subset X$		
Decision	$f(X) = \sum \theta_i(Y) \times a_i(Z)$	$Y \subset X$, $Z \subset X$	$Y \cap X = \varnothing$	
Spectral	$f(X) = \sum \theta_i(X) \times a_i$	$Y \subset X$, $Z \subset X$	$Y \cap X = \varnothing$	$Y = X$, $Z = \varnothing$

Bases

$$\Omega = \{+, \times, \theta_i \;(i = \overline{0, m-1})\}$$

Logic	Base	Set	Addition	Multiplication
Boolean	Boole, 1854	$\{0,1\}$	Disjunction $\vee$	Conjunction $\&$
	Zhegalkin, 1927	$\{0,1\}$	Nonequivalence $\oplus$	Conjunction $\&$
	Malyugin, Merekin, 1963	$\{..., -1, 0, 1, ...\}$	Arithmetic $+$	Conjunction $\&$
Multi-valued	Yablonskii, Post, 1958	$\{0, 1, \ldots, k-1\}$	Maximum max	Minimum min
	Dubrova, Musion, 1996	$\{0, 1, \ldots, k-1\}$	Addition mod k	Minimum min
	Malyugin, Vykhovanets, 1998	$\{..., -1, 0, 1, ...\}$	Arithmetic $+$	Digit-to-digit $\vee$, $\&$, $\oplus$
	Cohn, 1960	$\{0, 1, \ldots, p^n - 1\}$	Finite field $+$	Finite field $\times$
	Tosic, 1972	$\{..., -1, 0, 1, ...\}$	Arithmetic $+$	Arithmetic $\times$

Motivation

- We shall generalize and systematize the well-known forms of spectral decomposition in a wide class of bases defined by different addition and multiplication operations and spectral functions.
- The class of operations forming the base Ω and the corresponding type of spectral functions depend on the existence of methods for determining spectral coefficients a_i for known (given) values of functions.
- We shell also study generalized spectral forms and evaluate their effectiveness.

Definition. Discrete function

$$f(X),\ X=\{x_0, x_1, \ldots, x_{n-1}\}$$

$$\mathop{\times}_{j=0}^{n-1} N_{k_j} \xrightarrow{fun} N_{k_f}\ ,$$

$$N_k=\{0,1,\ldots,k-1\}$$

Example. Truth table

Variables	x	x_1	x_0	x_1	x_0	f	Function
Domain	N_6	N_2	N_3	N_3	N_2	N_4	Domain
	0	0	0	0	0	3	
	1	0	1	0	1	0	
	2	0	2	1	0	1	
	3	1	0	1	1	2	
	4	1	1	2	0	2	
	5	1	2	2	1	1	

Definition. Positional notation

$$f(x_0, x_1, \ldots, x_{n-1}) = f(x)$$

$$x_j=\overline{0,k_j-1},\ x=\overline{0,m-1},\ m=k_0k_1\ldots k_{n-1}$$

$$x=(x_{n-1}\ldots x_1\ x_0)_{k_{n-1}\ldots k_1k_0}$$

$$x=\sum_{j=0}^{n-1} x_j \prod_{t=0}^{j-1} k_t$$

Example. Positional notation

$x_0=2$, $k_0=3$	$x=(x_1\ x_0)_{2,3}=(1\,2)_{2,3}=1\cdot 3+2\cdot 1=5$
$x_1=1$, $k_1=2$	

Definition. Discrete operation

A discrete operation is a discrete function that strongly depends on its variables.

Generating Algebra

Let us consider the algebraic system $\boldsymbol{R} = \langle \boldsymbol{N}_{\boldsymbol{k}}, +, \times \rangle$ formed by two binary $\boldsymbol{k}$-valued operations and let us represent an function $\boldsymbol{f}(\boldsymbol{X}) = \boldsymbol{f}(x_0, x_1, \ldots, x_{n-1})$ of $\boldsymbol{n}$ variables in spectral form over algebra $\boldsymbol{R}$.

A. Variables Sets

$f(X) = f(X', X'')$,		$X' \cap X'' = \varnothing$.	
$X' \subseteq X$,	$k' = \prod_{(k_j \in X')} k_j$;	$X'' \subset X$,	$k'' = \prod_{(k_j \in X'')} k_j$.
Length of the vector of f is $m = k'k''$			

Example. Preparing of the variable sets

Function	$f(X) = [022030230231012113201010]$, $k_f = 4$, $m = 24$;			
Variables	$x_0 \in N_2$,	$x_1 \in N_2$,	$x_2 \in N_3$,	$x_3 \in N_2$;
Digits	$k_0 = 2$,	$k_1 = 2$,	$k_2 = 3$,	$k_3 = 2$;

Truth table

X	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
x_3	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1
x_2	0	0	0	0	1	1	1	1	2	2	2	2	0	0	0	0	1	1	1	1	2	2	2	2
x_1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
x_0	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
f	0	2	2	0	3	0	2	3	0	2	3	1	0	1	2	1	1	3	2	0	1	0	1	0

Sets	$X' = \{x_0, x_1\}$,	$X'' = \{x_2, x_3\}$;
Digits	$k' = k_0 k_1 = 4$,	$k'' = k_2 k_3 = 6$;

Truth table

		x_3	0	0	0	1	1	1
		x_2	0	1	2	0	1	2
x_1	x_0	X' \ X''	0	1	2	3	4	5
0	0	0	0	3	0	0	1	1
0	1	1	2	0	2	1	3	0
1	0	2	2	2	3	2	2	1
1	1	3	0	3	1	1	0	0

B. Spectral Expansion

Let us expand the function f in the algebra R by a system of functions θ_i

$$f(X', X'') = \sum_{i=0}^{k'-1} \theta_i(X') \times a_i(X'') \quad \textbf{(1)}$$

where θ_i are spectral functions, a_i are spectral coefficients (some functions)

Substituting the values of the variable X' into (1) we obtain equations

$$\begin{cases} f(0, X'') & = & \theta_0(0)\times a_0(X'') & +\ldots+ & \theta_{k'-1}(0)\times a_{k'-1}(X'') \\ f(1, X'') & = & \theta_0(1)\times a_0(X'') & +\ldots+ & \theta_{k'-1}(1)\times a_{k'-1}(X'') \\ \vdots & = & \vdots & \ddots & \vdots \\ f(k'-1, X'') & = & \theta_0(k'-1)\times a_0(X'') & +\ldots+ & \theta_{k'-1}(k'-1)\times a_{k'-1}(X'') \end{cases} \quad \textbf{(2)}$$

In matrix form, (2) is expressed as

$$\begin{bmatrix} f_0(X'') \\ f_1(X'') \\ \vdots \\ f_{k'-1}(X'') \end{bmatrix} = \begin{bmatrix} \theta_0(0) & \theta_1(0) & \cdots & \theta_{k'-1}(0) \\ \theta_0(1) & \theta_1(1) & \cdots & \theta_{k'-1}(1) \\ \vdots & \vdots & \ddots & \vdots \\ \theta_0(k'-1) & \theta_1(k'-1) & \cdots & \theta_{k'-1}(k'-1) \end{bmatrix} \text{x} \begin{bmatrix} a_0(X'') \\ a_1(X'') \\ \vdots \\ a_{k'-1}(X'') \end{bmatrix} \quad \textbf{(3)}$$

where $f_j(X'')$ are vectors of some subfunction of function f,
$a_j(X'')$ are vectors of the other function (spectral coefficients).

C. Solving the system of algebraic equations

If we can solve (2) for $a_j(X'')$, we will find a matrix Q

$$F(X'') = D \times A(X''), \quad A(X'') = Q \times F(X'') \quad \textbf{(4)}$$

where F (A) is a $k' \times k''$-matrix of the function f (of the coefficients),
D (Q) is square matrix of direct (inverse) transformation.

$$\begin{bmatrix} a_0(X'') \\ a_1(X'') \\ \vdots \\ a_{k'-1}(X'') \end{bmatrix} = \begin{bmatrix} \vartheta_0(0) & \vartheta_1(0) & \cdots & \vartheta_{k'-1}(0) \\ \vartheta_0(1) & \vartheta_1(1) & \cdots & \vartheta_{k'-1}(1) \\ \vdots & \vdots & \ddots & \vdots \\ \vartheta_0(k'-1) & \vartheta_1(k'-1) & \cdots & \vartheta_{k'-1}(k'-1) \end{bmatrix} \text{x} \begin{bmatrix} f_0(X'') \\ f_1(X'') \\ \vdots \\ f_{k'-1}(X'') \end{bmatrix} \quad \textbf{(5)}$$

where $\vartheta_i(X')$ are vectors of some functions.

Equation (3) shows that the columns of D are characteristic vector of spectral functions θ_i. In turn, equation (5) shows that the columns of Q can be regarded as vectors of orthogonal functions ϑ_i

There are several algebras that are aid in solving system (3) for a_j.

Algebra of Logic

Let us consider the algebra $R_L = \langle N_k, +, \times \rangle$ in which there exist elements σ (zero) and τ (unit) such that for all $a \in N_k$,

$a + \sigma = a$ $\sigma + a = a$	$\sigma \times a = \sigma$ $\tau \times a = a$

Example. Operations of the Algebra of Logic

$N_4 = \{0, 1, 2, 3\}$	$\sigma = 0,\ \tau = 3$
$+ = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & * & * & * \\ 2 & * & * & * \\ 3 & * & * & * \end{bmatrix}$	$\times = \begin{bmatrix} 0 & 0 & 0 & 0 \\ * & * & * & * \\ * & * & * & * \\ 0 & 1 & 2 & 3 \end{bmatrix}$

where * is indifferent (any element of N_4 may exist in place of *)

$0+0=0$	$0+0=0$	$0\times 0=0$	$3\times 0=0$
$1+0=1$	$0+1=1$	$0\times 1=0$	$3\times 1=1$
$2+0=2$	$0+2=2$	$0\times 2=0$	$3\times 2=2$
$3+0=3$	$0+3=3$	$0\times 3=0$	$3\times 3=3$

Definition. A Matrix of Permutations

A matrix of permutations $P_{k'}$ of order k' is a square matrix consisting of zeros and units and has exactly one unit each in every row (column).

Example. The Matrix of Permutation ($\sigma = 0,\ \tau = 3$)

$$P_3 = \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 3 \\ 3 & 0 & 0 \end{bmatrix}$$

Theorem. Decomposition over Algebra of Logic

An arbitrary function f is representable in Algebra of Logic R_L by spectral expansion if D is a matrix of permutations. Then $Q = D^T$ and $Q \times D = E$, where E is a unit matrix and T is the transpose.

Note: The number of permutation matrix (or the number of different representation for the function f over R_L) is $N_L(k') = k'!$

Example. An expansion of function over Algebra of Logic

Algebra	$R_L = \langle N_4, +, \times \rangle$,	$N_4 = \{0, 1, 2, 3\}$;
Constants	$\sigma = 0$,	$\tau = 3$;
Operations	$+ = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & * & * & * \\ 2 & * & * & * \\ 3 & * & * & * \end{bmatrix}$,	$\times = \begin{bmatrix} 0 & 0 & 0 & 0 \\ * & * & * & * \\ * & * & * & * \\ 0 & 1 & 2 & 3 \end{bmatrix}$;
Function	$f(X) = [3\,0\,1\,2\,2\,1]$; $k_f = 4$, $m = 6$;	

Truth table

x_1	0	0	0	1	1	1
x_0	0	1	2	0	1	2
f	3	0	1	2	2	1

Sets and digits	$X' = \{x_0\}$, $k' = 3$;	$X'' = \{x_1\}$, $k'' = 2$;

Truth table

X'' \ X'	0	1	2
0	3	0	1
1	2	2	1

Expansion	$f(X', X'') = \theta_0(X') \times a_0(X'') + \theta_1(X') \times a_1(X'') + \theta_2(X') \times a_2(X'')$	
Spectral functions	$D = \begin{bmatrix} \theta_0(0) & \theta_1(0) & \theta_2(0) \\ \theta_0(1) & \theta_1(1) & \theta_2(1) \\ \theta_0(2) & \theta_1(2) & \theta_2(2) \end{bmatrix}$	$Q = \begin{bmatrix} \vartheta_0(0) & \vartheta_1(0) & \vartheta_2(0) \\ \vartheta_0(1) & \vartheta_1(1) & \vartheta_2(1) \\ \vartheta_0(2) & \vartheta_1(2) & \vartheta_2(2) \end{bmatrix}$
Transforms	$D = \begin{bmatrix} 0 & 0 & 3 \\ 3 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}$	$Q = D^T = \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 3 \\ 3 & 0 & 0 \end{bmatrix}$
Vectors	$F(X'') = \begin{bmatrix} f(0, X'') \\ f(1, X'') \\ f(2, X'') \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}$	$A(X'') = \begin{bmatrix} a_0(X'') \\ a_1(X'') \\ a_2(X'') \end{bmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \\ ? & ? \end{bmatrix}$
Coefficients	$A(X'') = Q \times F(X'') = \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 3 \\ 3 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 3 & 2 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \\ 3 & 2 \end{bmatrix}$	
Expansion	$f(X', X'') = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}(X') \times \begin{bmatrix} 0 \\ 2 \end{bmatrix}(X'') + \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}(X') \times \begin{bmatrix} 1 \\ 1 \end{bmatrix}(X'') + \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}(X') \times \begin{bmatrix} 3 \\ 2 \end{bmatrix}(X'')$	
where $[y_i](x) = y_x$ is the unary operation defined by vector $[y_0, y_1, \dots, y_{k-1}]$		
Test	$f(X) = f(X', X'')_{k'k''}$,	$f(5) = f(2,1)_{3,2} = 0 \times 2 + 3 \times 1 + 0 \times 2 = 1$

Multiplicative Algebra

Let us determine an algebra $\boldsymbol{R_M} = \langle \boldsymbol{N_k}, +, \times \rangle$ in which operations satisfy the condition of the algebra $\boldsymbol{R_L}$ and multiplication additionally forms a group $\boldsymbol{G_M} = \langle \boldsymbol{N_k} \setminus \{\sigma\}, \times \rangle$ on the set $\boldsymbol{N_k}$, expect for the zero element $\boldsymbol{\sigma}$.

$\boldsymbol{a} + \boldsymbol{\sigma} = \boldsymbol{a}$	$\boldsymbol{\sigma} + \boldsymbol{a} = \boldsymbol{a}$	$\boldsymbol{G_M} = \langle \boldsymbol{N_k} \setminus \{\sigma\}, \times \rangle$ is a group
$\boldsymbol{\sigma} \times \boldsymbol{a} = \boldsymbol{\sigma}$	$\boldsymbol{\tau} \times \boldsymbol{a} = \boldsymbol{a}$	

Definition. A Group

A group $\boldsymbol{G} = \langle \boldsymbol{N}, \times \rangle$ defined on the set $\boldsymbol{N}$ is a algebraic system such that

1) the operation $\times$ is associative, or for all $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \boldsymbol{N}$, $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = \boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$;
2) equations $\boldsymbol{a} \times \boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{x} \times \boldsymbol{a} = \boldsymbol{b}$ is uniquely solvable for $\boldsymbol{x}$ with any $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{N}$.

Note: Every element $\boldsymbol{a} \in \boldsymbol{N}$ has inverse elements $\boldsymbol{a}^{-1}$ such that $\boldsymbol{a} \times \boldsymbol{a}^{-1} = \boldsymbol{\tau}$ and $\boldsymbol{a}^{-1} \times \boldsymbol{a} = \boldsymbol{\tau}$ where $\boldsymbol{\tau}$ also is called a neutral element of group.

Example. The Group $\boldsymbol{G_M} = \langle N_4 \setminus \{\sigma\}, \times \rangle$ ($\boldsymbol{\sigma} = 0$, $\boldsymbol{\tau} = 3$)

$\boldsymbol{N} = \{1, 2, 3\}$	$\times = \left[\begin{array}{c\|ccc} * & * & * & * \\ \hline * & 2 & 3 & 1 \\ * & 3 & 1 & 2 \\ * & 1 & 2 & 3 \end{array}\right]$

Note: A group has the matrix of operation such that in any row (column) there are all elements of $\boldsymbol{N}$.

Definition. A Monomial Matrix

A monomial matrix $\boldsymbol{M_{k'}}$ of order $\boldsymbol{k'}$ is a square matrix obtained through permutation of rows (columns) of the diagonal matrix elements $\boldsymbol{d_i} \neq \boldsymbol{\sigma}$ $(\boldsymbol{i} = \overline{0, \boldsymbol{k'} - 1})$.

Example. The Monomial Matrix ($\boldsymbol{\sigma} = 0$)

Diagonal matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$	Monomial matrix $\boldsymbol{M}_3 = \begin{bmatrix} 0 & 0 & 1 \\ 3 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$

Theorem. Decomposition over Multiplicative Algebra

An arbitrary function f is representable in Multiplicative Algebra $\boldsymbol{R_M}$ by spectral expansion if $\boldsymbol{D}$ is a monomial matrix. Then $\boldsymbol{Q} = \tilde{\boldsymbol{D}}^{\boldsymbol{T}}$ and $\boldsymbol{Q} \times \boldsymbol{D} = \boldsymbol{E}$, where $\tilde{\boldsymbol{D}}$ is obtained from $\boldsymbol{D}$ upon replacement of nonzero elements $\boldsymbol{a}$ by their inverse elements $\boldsymbol{a}^{-1}$ in the group $\boldsymbol{G_M}$.

Note: Every function in $\boldsymbol{R_M}$ has exactly $\boldsymbol{N_M}(\boldsymbol{k'}) = \boldsymbol{k'}!(\boldsymbol{k'} - 1)^{\boldsymbol{k'}}$ representations for fixed $\boldsymbol{k'}$.

Example. An expansion over Multiplicative Algebra

<table>
<tr><td>Algebra</td><td colspan="2">$\boldsymbol{R_M} = \langle \boldsymbol{N}_4, +, \times \rangle$,</td><td>$\boldsymbol{N}_4 = \{0, 1, 2, 3\}$;</td></tr>
<tr><td>Constants</td><td colspan="2">$\boldsymbol{\sigma} = 0$,</td><td>$\boldsymbol{\tau} = 3$;</td></tr>
<tr><td>Operations</td><td colspan="2">$+ = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & * & * & * \\ 2 & * & * & * \\ 3 & * & * & * \end{bmatrix}$,</td><td>$\times = \begin{bmatrix} 0 & 0 & 0 & 0 \\ * & 2 & 3 & 1 \\ * & 3 & 1 & 2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$;</td></tr>
<tr><td>Inverse elements</td><td>$1^{-1} = 2$,</td><td>$2^{-1} = 1$,</td><td>$3^{-1} = 3$;</td></tr>
<tr><td>Function</td><td colspan="3">$\boldsymbol{f}(\boldsymbol{X}) = [3\,0\,1\,2\,2\,1]$; $\boldsymbol{k_f} = 4$, $\boldsymbol{m} = 6$;</td></tr>
<tr><td>Sets and digits</td><td colspan="2">$\boldsymbol{X'} = \{x_0\}$, $\boldsymbol{k'} = 3$;</td><td>$\boldsymbol{X''} = \{x_1\}$, $\boldsymbol{k''} = 2$;</td></tr>
<tr><td>Truth table</td><td colspan="3"><table>
<tr><td>X'' \ X'</td><td>0</td><td>1</td><td>2</td></tr>
<tr><td>0</td><td>3</td><td>0</td><td>1</td></tr>
<tr><td>1</td><td>2</td><td>2</td><td>1</td></tr>
</table></td></tr>
<tr><td>Construction</td><td colspan="3">$\boldsymbol{f}(\boldsymbol{X'}, \boldsymbol{X''}) = \boldsymbol{\theta}_0(\boldsymbol{X'}) \times \boldsymbol{a}_0(\boldsymbol{X''}) + \boldsymbol{\theta}_1(\boldsymbol{X'}) \times \boldsymbol{a}_1(\boldsymbol{X''}) + \boldsymbol{\theta}_2(\boldsymbol{X'}) \times \boldsymbol{a}_2(\boldsymbol{X''})$;</td></tr>
<tr><td>Transforms</td><td colspan="2">$\boldsymbol{D} = \begin{bmatrix} 0 & 0 & 1 \\ 3 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$,</td><td>$\boldsymbol{Q} = \tilde{\boldsymbol{D}}^{\boldsymbol{T}} = \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix}$;</td></tr>
<tr><td>Coefficients</td><td colspan="3">$\boldsymbol{A}(\boldsymbol{X''}) = \boldsymbol{Q} \times \boldsymbol{F}(\boldsymbol{X''}) = \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 3 & 2 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & 2 \\ 2 & 1 \end{bmatrix}$;</td></tr>
<tr><td>Expansion</td><td colspan="3">$\boldsymbol{f}(\boldsymbol{X'}, \boldsymbol{X''}) = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}(\boldsymbol{X'}) \times \begin{bmatrix} 0 \\ 2 \end{bmatrix}(\boldsymbol{X''}) + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}(\boldsymbol{X'}) \times \begin{bmatrix} 2 \\ 2 \end{bmatrix}(\boldsymbol{X''}) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}(\boldsymbol{X'}) \times \begin{bmatrix} 2 \\ 1 \end{bmatrix}(\boldsymbol{X''})$;</td></tr>
<tr><td>Test</td><td colspan="3">$\boldsymbol{f}(5) = \boldsymbol{f}(2,1)_{3,2} = 0 \times 2 + 2 \times 2 + 0 \times 1 = 1$</td></tr>
</table>

Additive Algebra

Let $R_A = \langle N_k, +, \times \rangle$, where the operations of addition and multiplication satisfy the condition of the algebra R_L and addition forms a commutative (Abelian) group $G_A = \langle N_k, + \rangle$, i.e. every element $a \in N_k$ has negative element $-a$ such that $a + (-a) = \sigma$.

$a + \sigma = a$	$\sigma + a = a$	$G_A = \langle N_k, + \rangle$ is an Abelian group
$\sigma \times a = \sigma$	$\tau \times a = a$	

Definition. An Order of element

The order of an element a of the group G_A is the minimal integer $c_a > 0$ for which

$$c_a \circ a = \underbrace{a + a + \ldots + a}_{c_a} = \sigma .$$

Let $0 \circ a = \sigma$ and $(-\lambda) \circ a = \lambda \circ (-a)$ for any $a \in N_k$ and $\lambda \in Z$, where Z is integers.

Definition. A Cyclic Order of group

The cyclic order of a group G_A is the minimal order of all elements, except for σ:

$$C_A = \min_{(a \neq \sigma)} c_a .$$

Example. The Cyclic Order of group ($\sigma = 0$)

$G_A = \langle N_4, + \rangle$, $N_4 = \{0, 1, 2, 3\}$; $c_1 = c_2 = c_3 = 2$ as $a + a = \sigma$; $C_A = 2$;	$+ = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix}.$

Definition. A Logical Matrix

A logical matrix $L_{k'}$ of order k' is a square matrix consisting of zeros and units of the algebra R_A. The adjoint of a logical matrix $L_{k'}$ is a matrix $\hat{L}_{k'}$, which is obtained from $L_{k'}$ upon replacements of zeros and units elements of the algebra R_A by zeros and units of the ring of integers $R_Z = \langle Z, +, \times \rangle$.

Example. The Logical Matrix ($\sigma = 0$, $\tau = 3$)

$L_3 = \begin{bmatrix} 3 & 3 & 0 \\ 3 & 0 & 3 \\ 0 & 3 & 3 \end{bmatrix}$	$\hat{L}_3 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Theorem. Decomposition over Multiplicative Algebra

Every function $\boldsymbol{f}$ is representable in additive algebra $\boldsymbol{R_A}$ by spectral expansion if $\boldsymbol{D}$ is a logical matrix and the modulus of the determinant Δ of the adjoint matrix $\hat{\boldsymbol{D}}$ is not zero and less then the cyclic order of the additive group $\boldsymbol{G_A}$. Then $\Delta \circ \boldsymbol{A} = \overline{\boldsymbol{D}}^{\boldsymbol{T}} \circ \boldsymbol{F}$, where $\overline{\boldsymbol{D}}$ is the matrix of algebraic complements computed in the ring of integers $\boldsymbol{R_Z}$.

Example. An expansion over Multiplicative Algebra

<table>
<tr><td>Algebra</td><td colspan="3">$\boldsymbol{R_A} = \langle \boldsymbol{N}_4, +, \times \rangle$,</td><td colspan="2">$\boldsymbol{N}_4 = \{0, 1, 2, 3\}$;</td></tr>
<tr><td>Constants</td><td colspan="3">$\boldsymbol{\sigma} = 0$,</td><td colspan="2">$\boldsymbol{\tau} = 3$;</td></tr>
<tr><td>Operations</td><td colspan="3">$+ = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix}$,</td><td colspan="2">$\times = \begin{bmatrix} 0 & 0 & 0 & 0 \\ * & * & * & * \\ * & * & * & * \\ 0 & 1 & 2 & 3 \end{bmatrix}$;</td></tr>
<tr><td>Negative elements</td><td>$-1 = 1$,</td><td colspan="3">$-2 = 2$,</td><td>$-3 = 3$;</td></tr>
<tr><td>Function</td><td colspan="5">$\boldsymbol{f}(\boldsymbol{X}) = [3\,0\,1\,2\,2\,1]$; $\boldsymbol{k_f} = 4$, $\boldsymbol{m} = 6$;</td></tr>
<tr><td>Sets and digits</td><td colspan="3">$\boldsymbol{X'} = \{x_0\}$, $\boldsymbol{k'} = 3$;</td><td colspan="2">$\boldsymbol{X''} = \{x_1\}$, $\boldsymbol{k''} = 2$;</td></tr>
<tr><td>Truth table</td><td colspan="5"><table>
<tr><td>$\boldsymbol{X''}$ \ $\boldsymbol{X'}$</td><td>0</td><td>1</td><td>2</td></tr>
<tr><td>0</td><td>3</td><td>0</td><td>1</td></tr>
<tr><td>1</td><td>2</td><td>2</td><td>1</td></tr>
</table></td></tr>
<tr><td>Construction</td><td colspan="5">$\boldsymbol{f}(\boldsymbol{X'}, \boldsymbol{X''}) = \boldsymbol{\theta}_0(\boldsymbol{X'}) \times \boldsymbol{a}_0(\boldsymbol{X''}) + \boldsymbol{\theta}_1(\boldsymbol{X'}) \times \boldsymbol{a}_1(\boldsymbol{X''}) + \boldsymbol{\theta}_2(\boldsymbol{X'}) \times \boldsymbol{a}_2(\boldsymbol{X''})$;</td></tr>
<tr><td>Transforms</td><td>$\boldsymbol{D} = \begin{bmatrix} 3 & 3 & 0 \\ 3 & 0 & 3 \\ 0 & 3 & 3 \end{bmatrix}$,</td><td>$\hat{\boldsymbol{D}} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$,</td><td colspan="2">$\Delta = -1$,</td><td>$\hat{\boldsymbol{D}} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix}$</td></tr>
<tr><td>Coefficients</td><td colspan="5">$\Delta \circ \boldsymbol{A}(\boldsymbol{X''}) = \overline{\boldsymbol{D}}^{\boldsymbol{T}} \circ \boldsymbol{F}(\boldsymbol{X''}) = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix} \circ \begin{bmatrix} 3 & 2 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}$; $\boldsymbol{A} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}$</td></tr>
<tr><td>Expansion</td><td colspan="5">$\boldsymbol{f}(\boldsymbol{X'}, \boldsymbol{X''}) = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}(\boldsymbol{X'}) \times \begin{bmatrix} 2 \\ 3 \end{bmatrix}(\boldsymbol{X''}) + \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}(\boldsymbol{X'}) \times \begin{bmatrix} 1 \\ 1 \end{bmatrix}(\boldsymbol{X''}) + \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}(\boldsymbol{X'}) \times \begin{bmatrix} 2 \\ 1 \end{bmatrix}(\boldsymbol{X''})$;</td></tr>
<tr><td>Test</td><td colspan="5">$\boldsymbol{f}(5) = \boldsymbol{f}(2,1)_{3,2} = 0 \times 3 + 3 \times 1 + 0 \times 1 = 1$</td></tr>
</table>

Finite Field

Let $R_F = \langle N_k, +, \times \rangle$, where the operations of addition and multiplication form a field on the set N_k and thereby satisfy the condition of all three algebras R_L, R_M and R_A.

$a + \sigma = a$	$\sigma \times a = \sigma$	$G_A = \langle N_k, + \rangle$ is a Abelian group
$\sigma + a = a$	$\tau \times a = a$	$G_M = \langle N_k \setminus \{\sigma\}, \times \rangle$ is a group

Note: It is well known that a finite field R_F defined up to a isomorphism exists for $k = p^q$, where p is prime, q is an integer.

Theorem. Decomposition over Finite Field

A function f is representable in finite field R_F by spectral expansion if determinant of matrix D in the field is nonzero. Then $Q = D^{-1}$ and $Q \times D = E$, where D^{-1} is the inverse of the matrix D over R_F.

Note: The number of invertible matrices (number of spectral representatives) in a field

$$N_F(k, k') = k^{k'(k'-1)/2} \prod_{i=1}^{k'} \left(k^i - 1\right).$$

Example. An expansion over Additive Algebra

Algebra	$R_F = \langle N_4, +, \times \rangle$,	$N_4 = \{0, 1, 2, 3\}$,	$\sigma = 0$, $\tau = 3$;
Operations	$+ = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix}$,	$\times = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 1 \\ 0 & 3 & 1 & 2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$;	
Function	$f(X) = [3\,0\,1\,2\,2\,1]$;		
Sets and digits	$X' = \{x_0\}$, $k' = 3$;	$X'' = \{x_1\}$, $k'' = 2$;	
Transforms	$D = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 0 \\ 0 & 2 & 3 \end{bmatrix}$,	$D^{-1} = \begin{bmatrix} 1 & 1 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 2 \end{bmatrix}$,	$A = \begin{bmatrix} 1 & 1 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 2 \end{bmatrix} \times \begin{bmatrix} 3 & 2 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 2 \\ 1 & 0 \end{bmatrix}$;
Expansion	$f(X', X'') = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}(X') \times \begin{bmatrix} 0 \\ 1 \end{bmatrix}(X'') + \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}(X') \times \begin{bmatrix} 0 \\ 2 \end{bmatrix}(X'') + \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}(X') \times \begin{bmatrix} 1 \\ 0 \end{bmatrix}(X'')$;		
Test	$f(5) = f(2,1)_{3,2} = 0 \times 1 + 2 \times 2 + 3 \times 0 = 1$		

Integral Ring

Let us consider the algebra $\boldsymbol{R_R} = \langle \boldsymbol{N_k}, +, \times \rangle$, where the operations of addition and multiplication form a ring on the set $\boldsymbol{N_k}$

Definition. A Ring

A ring $\boldsymbol{R_R} = \langle \boldsymbol{N_k}, +, \times \rangle$ defined on the set $\boldsymbol{N_k}$ is an algebraic system such that

1) the addition generates a commutative (Abelian) group on the set $\boldsymbol{N_k}$;
2) distributive law holds for the multiplication relative the addition; for all $\boldsymbol{a,b,c} \in \boldsymbol{N_k}$

$$\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = (\boldsymbol{a} \times \boldsymbol{b}) + (\boldsymbol{a} \times \boldsymbol{c}) \text{ and } (\boldsymbol{a} + \boldsymbol{b}) \times \boldsymbol{c} = (\boldsymbol{a} \times \boldsymbol{c}) + (\boldsymbol{b} \times \boldsymbol{c}).$$

Note: Our system of algebraic equations can be solved over the ring $\boldsymbol{R_R}$ if and only if $\boldsymbol{R_R}$ is an integral domain (commutative ring with unity and without zero divisors).

If the number of elements $\boldsymbol{k}$ is finite, then the integral ring $\boldsymbol{R_R}$ is isomorphic to a finite field (we have already examined this case).

Let $\boldsymbol{R_R} = \langle \boldsymbol{N}_0, +, \times \rangle$, where $\boldsymbol{N}_0 = \{\ldots, -1, 0, 1, \ldots\}$ and the addition and the multiplication are arithmetic operations (any integral ring defined on the set $\boldsymbol{N}_0$ is isomorphic to $\boldsymbol{R_R}$).

Theorem. Decomposition over Integral Ring

A function $\boldsymbol{f}$ is representable in integral ring $\boldsymbol{R_R}$ by spectral expansion if determinant Δ of matrix $\boldsymbol{D}$ in the ring $\boldsymbol{R_R}$ is nonzero. Then $\Delta \times \boldsymbol{A} = \overline{\boldsymbol{D}}^{\boldsymbol{T}} \times \boldsymbol{F}$, where $\overline{\boldsymbol{D}}$ is the matrix of algebraic complements computed in the ring $\boldsymbol{R_R}$.

Example. An expansion over Integral Ring

Algebra	$\boldsymbol{R_R} = \langle \boldsymbol{N}_0, +, \times \rangle$,	$\boldsymbol{N}_0 = \{\ldots, -1, 0, 1, \ldots\}$,	$\boldsymbol{\sigma} = 0$, $\boldsymbol{\tau} = 1$;
Function	$\boldsymbol{f}(\boldsymbol{X}) = [3\,0\,1\,2\,2\,1]$;		
Sets and digits	$\boldsymbol{X}' = \{x_0\}$, $\boldsymbol{k}' = 3$;		$\boldsymbol{X}'' = \{x_1\}$, $\boldsymbol{k}'' = 2$;
Transforms	$\boldsymbol{D} = \begin{bmatrix} 3 & -2 & 1 \\ 1 & 0 & 2 \\ 4 & -1 & 3 \end{bmatrix}$,	$\boldsymbol{A} = -\frac{1}{5}\begin{bmatrix} 2 & 5 & -4 \\ 5 & 5 & -5 \\ -1 & 5 & 2 \end{bmatrix} \times \begin{bmatrix} 3 & 2 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} 2 & 10 \\ 10 & 15 \\ -1 & -10 \end{bmatrix}$;	
Expansion	$\boldsymbol{f}(\boldsymbol{X}', \boldsymbol{X}'') = -\frac{1}{5}\left\{\begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}(\boldsymbol{X}') \times \begin{bmatrix} 2 \\ 10 \end{bmatrix}(\boldsymbol{X}'') + \begin{bmatrix} -2 \\ 0 \\ -1 \end{bmatrix}(\boldsymbol{X}') \times \begin{bmatrix} 10 \\ 15 \end{bmatrix}(\boldsymbol{X}'') + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}(\boldsymbol{X}') \times \begin{bmatrix} -1 \\ -10 \end{bmatrix}(\boldsymbol{X}'')\right\}$;		
Test	$\boldsymbol{f}(5) = \boldsymbol{f}(2,1)_{3,2} = -\frac{1}{5}(4 \times 10 + (-1) \times 15 + 3 \times (-10)) = 1$		

Polynomial Forms

Let us find the classes of spectral functions $\theta_i(X')$ having a compact representation. Polynomial forms having a homogeneous analytical construction of spectral functions

$$\theta_i(X') = x_0^{i_0} \bullet_0 x_1^{i_1} \bullet_1 \dots \bullet_{n-2} x_{n-1}^{i_{n-1}} \bullet_{n-1} c_i \qquad (i = \overline{0, k'-1}),$$

where $X' = \{x_0, x_1, \dots, x_{n-1}\}$ is a variable set;

c_i are arbitrary constants;

$\bullet_j$ are connective functions of two variables (may be differences);

$x_j^{i_j} = x_j \circ_j i_j$ are power functions $\circ_j$ defined for each variable x_j separately;

$i = (i_{n-1} \dots i_1\, i_0)_{k_{n-1} \dots k_1 k_0}$ is the representation of the number i of spectral function $\theta_i(X')$ in positional system with bases $k_{n-1}, \dots, k_1, k_0$.

Example. The Polynomial Form

Variables	$X' = \{x_0, x_1, x_2\}$,	$K' = \{2,3,2\}$,	$k' = 2 \cdot 3 \cdot 2 = 12$;
Construction	$\theta_i(X') = x_0^{i_0} \bullet_0 x_1^{i_1} \bullet_1 x_2^{i_2} \bullet_2 c_i$		
Connectives	$\bullet_0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \&$,	$\bullet_1 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \div$,	$\bullet_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \neq$;
Powers	$\circ_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \approx$	$\circ_1 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix} = \oplus$	$\circ_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \uparrow$
Function 0	$0 = (000)_{2,3,2}$,	$c_0 = 0$,	$\theta_0(X') = x_0^0 \,\&\, x_1^0 \div x_2^0 \neq c_0 = \bar{x}_0 \,\&\, x_1 \div 1$;
Function 1	$1 = (001)_{2,3,2}$	$c_1 = 1$,	$\theta_1(X') = x_0^1 \,\&\, x_1^0 \div x_2^0 \neq c_1 = \overline{x_0 \,\&\, x_1 \div 1}$;
...	...	...	...
Function 11	$11 = (121)_{2,3,2}$	$c_{11} = 0$,	$\theta_{11}(X') = x_0^1 \,\&\, x^2 \div x_2^1 \neq c_{11} = x_0 \,\&\, (x_1 \oplus 2) \div x_2$.

Theorem. Representable in Polynomial Form

For a discrete function to be representable in polynomial form in a generating algebra R, it is necessary that the power function $\circ_j$ be essential operations in the domain $N_{k_j} \times N_{k_j}$ and the connective function $\bullet_j$ be operations in the domains of the values of $\circ_j$ and $\circ_{j+1}$.

Note: The matrix of a nonessential operation contains not less than two identical rows (columns). The matrix of an operation contains at least one row (column) that is different from other rows (columns)

Example. The well-known Polynomial Forms

Boolean

Form	R	$\bullet_j$	$\circ_j$
Conjunctive	$\langle N_2, \vee, \& \rangle$	$\&$	$\oplus$
Zhegalkin	$\langle N_2, \oplus, \& \rangle$	$\&$	$\oplus$
Arithmetical	$\langle N_0, +, \times \rangle$	$\&$	x^i
Hadamard	$\langle N_0, +, \times \rangle$	$\times$	$(-1)^{x \oplus i}$

Multi-valued

Form	R	$\bullet_j$	$\circ_j$
Yablonskii	$\langle N_k, \max, \min \rangle$	min	$x^i = \begin{cases} k-1, i = x \\ 0, i \neq x \end{cases}$
Galois	$\langle N_{p^q}, +, \times \rangle$	$\times$	x^i (over R)
Arithmetical	$\langle N_0, +, \times \rangle$	$\times$	x^i (over R)
Generalized	$\langle N_0, +, \times \rangle$	$\times$	$(-1)^{x+i (\mathrm{mod}\ p)}$

Synthesis of Polynomial Forms

Let us generalize the methods of synthesis of polynomial forms in algebra R of generating operations. We shall reduce the synthesis to constructing a matrix D with regard for the analytical structure of spectral functions and to finding the matrix $A(X'')$ for every matrix of function $F(X'')$.

The matrix D can be computed by the recurrent rule

$$D_0 = \Gamma_0, \qquad D_{j+1} = D_j \otimes_j \Gamma_{j+1} \quad \left(j = \overline{0, n-2}\right), \qquad D = D_{n-1} \bullet_{n-1} C;$$

where Γ_j is the matrix of the power operation $\circ_j$;

$\otimes_j$ is the Kronecker product relative to the operation $\bullet_j$;

C is a matrix consisting of k' identical rows of k' arbitrary constants

Definition. Kronecker Product

Let a binary operation $\bullet$, an $n_0 \times m_0$-matrix $A = \lfloor a_{i_0 j_0} \rfloor$ and an $n_1 \times m_1$-matrix $B = \lfloor b_{i_1 j_1} \rfloor$ be given. The Kronecker product of the matrix A and the matrix B relative to the operation $\bullet$ is the matrix $C = A \otimes_\bullet B$ with elements $c_{ij} = a_{i_0 j_0} \bullet a_{i_1 j_1}$, where $i = (i_1 i_0)_{n_1, n_0}$ and $j = (j_1 j_0)_{m_1, m_0}$ is representation i and j in positional system.

Note: Definition implies that C is a block matrix consisting on $n_1 \times m_1$ submatrices $C_{i_1 j_1} = A \bullet b_{i_1 j_1}$ of dimension $n_0 \times m_0$.

Example. The synthesis of polynomial form

Let us represent the function $f(X) = [301221]$ in polynomial form.

Algebra	$R_M = \langle N_4, +, \times \rangle;$
Operations	$+ = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & * & * & * \\ 2 & * & * & * \\ 3 & * & * & * \end{bmatrix},$ $\quad \times = \begin{bmatrix} 0 & 0 & 0 & 0 \\ * & 2 & 3 & 1 \\ * & 3 & 1 & 2 \\ 0 & 1 & 2 & 3 \end{bmatrix};$
Variables	$X = \{x_0, x_1\},$ $\quad K = \{3, 2\};$
Sets	$X' = \{x_0, x_1\},$ $\quad k' = 6;$ $\quad X'' = \varnothing,$ $\quad k'' = 1;$
Construction	$\theta_i(X') = x_0^{i_0} \bullet x_1^{i_1};$
Matrices	$\Gamma_0 = \begin{bmatrix} 2 & 3 & 2 \\ 2 & 2 & 0 \\ 1 & 2 & 2 \end{bmatrix},$ $\quad \Gamma_1 = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix},$ $\quad \bullet = \begin{bmatrix} 3 & 0 & 1 & 2 \\ 2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & 2 & 2 \end{bmatrix};$
Synthesis	$D = \Gamma_0 \otimes_{\bullet} \Gamma_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ $\quad Q = \tilde{D}^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$
Coefficients	$A = Q \times F = [333130]$
Formula	$f(x_0, x_1) = x_0^0 \bullet x_1^0 + x_0^1 \bullet x_1^0 + x_0^2 \bullet x_1^0 + (x_0^0 \bullet x_1^1) \times 1 + x_0^1 \bullet x_1^1$
Test	$f(2,1) = 2^0 \bullet 1^0 + 2^1 \bullet 1^0 + 2^2 \bullet 1^0 + (2^0 \bullet 1^1) \times 1 + 2^1 \bullet 1^1$
	$f(2,1) = 1 \bullet 3 + 2 \bullet 3 + 2 \bullet 3 + (1 \bullet 1) \times 1 + 2 \bullet 1$
	$f(2,1) = 1 + 0 + 0 + 0 \times 1 + 0 = 1$

Nonpolynomial Forms

Representation of discrete functions in Conjunctive, Disjunctive, Zhegalkin, Walsh, Hadamard, Haar, and the other forms find extensive application in practice.

The spectral function for some of them can be found as polynomial analytical constructions, for example, for the Conjunctive, Disjunctive, Zhegalkin and Hadamard forms. But there are forms that have no polynomial representations. The Haar and Walsh are typical nonpolynomial forms.

Let us examine the spectral features of nonpolynomial representations of discrete functions. In this case the spectral functions $\boldsymbol{\theta}(\boldsymbol{X'})$ have the analytical construction

$$\boldsymbol{\theta_i}(\boldsymbol{X'}) = \left(\sim_0^{(i)} \boldsymbol{x}_0\right) \bullet_0^{(i)} \left(\sim_1^{(i)} \boldsymbol{x}_1\right) \bullet_0^{(i)} \ldots \bullet_0^{(i)} \left(\sim_1^{(i)} \boldsymbol{x}_1\right) \bullet_0^{(i)} \boldsymbol{c}^{(i)}$$

where $\boldsymbol{X'} = \{\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{\boldsymbol{n}-1}\}$ is a variable set;

$\boldsymbol{c}^{(i)}$ are arbitrary constants;

$\bullet_j^{(i)}$ are binary functions (may be differences);

$\sim_j^{(i)}$ are unary functions (may be differences also);

Note: Unlike polynomial form, the operations in nonpolynomial construction depend on the index of spectral function (every spectral function $\boldsymbol{\theta_i}$ has specific set of operations).

Synthesis of Nonpolynomial Forms

Synthesis of nonpolynomial forms consists of determining spectral functions, which in the algebra of generating operations $\boldsymbol{R}$ may form columns of the direct transformation matrix $\boldsymbol{D}$.

Total number of such matrices in the algebras was computed early. The number of spectral representations is maximal in finite fields and integral rings. Obviously, polynomial forms are the particular case of nonpolynomial forms.

Example. The synthesis of nonpolynomial form

Let us construct a nonpolynomial form over the additive algebra.

Algebra	$\boldsymbol{R_A} = \langle \boldsymbol{N}_4, +, \times \rangle;$
Operations	$+ = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix},$ $\quad \times = \begin{bmatrix} 0 & 0 & 0 & 0 \\ * & * & * & * \\ * & * & * & * \\ 0 & 1 & 2 & 3 \end{bmatrix};$
Variables	$\boldsymbol{X'} = \{x_0, x_1\},$ $\quad \boldsymbol{K'} = \{3, 2\};$
Construction	$\boldsymbol{\theta_i}(\boldsymbol{X'}) = \sim_0^{(i)} \boldsymbol{x}_0 \bullet^{(i)} \sim_1^{(i)} \boldsymbol{x}_1;$
Operations	$\sim = \begin{bmatrix} \sigma \\ \sigma \\ \tau \end{bmatrix},$ $\quad \wedge = \begin{bmatrix} \tau \\ \sigma \\ \sigma \end{bmatrix},$ $\quad - = \begin{bmatrix} \tau \\ \sigma \end{bmatrix},$ $\quad \& = \begin{bmatrix} \sigma & \sigma \\ \sigma & \tau \end{bmatrix},$ $\quad \oplus = \begin{bmatrix} \sigma & \tau \\ \tau & \sigma \end{bmatrix};$
Functions	$\boldsymbol{D} = [\tau \quad x_0 \quad \tilde{x}_1 \quad \hat{x}_1 \quad x_0 \,\&\, \tilde{x}_1 \quad x_0 \oplus \hat{x}_1];$
Transform	$\boldsymbol{D} = \begin{array}{cc} x_1 & x_0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 2 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{array} \begin{bmatrix} \tau & \sigma & \sigma & \tau & \sigma & \sigma \\ \tau & \sigma & \sigma & \sigma & \sigma & \tau \\ \tau & \sigma & \tau & \sigma & \sigma & \tau \\ \tau & \tau & \sigma & \tau & \sigma & \tau \\ \tau & \tau & \sigma & \sigma & \sigma & \sigma \\ \tau & \tau & \tau & \sigma & \tau & \sigma \end{bmatrix},$ $\quad \Delta = 1,$ $\quad \overline{\boldsymbol{D}}^{\boldsymbol{T}} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix};$
	$\boldsymbol{A}(\boldsymbol{X''}) = \overline{\boldsymbol{D}}^{\boldsymbol{T}} \circ \boldsymbol{F}(\boldsymbol{X''}).$

Asymptotic estimates

Theorem. Asymptotic estimates

There exist methods of synthesizing a formal representation for discrete functions in which the number $\boldsymbol{M}$ of nonzero coefficients of a form and the number $\boldsymbol{L}$ of operations satisfy the asymptotic estimates

$$\boldsymbol{M}(\boldsymbol{m}) \sim \frac{4}{\boldsymbol{k}^4} \frac{\boldsymbol{k}^{\boldsymbol{n}}}{\boldsymbol{n}^2}, \; \boldsymbol{L}(\boldsymbol{m}) \sim \frac{8}{\boldsymbol{k}^4} \frac{\boldsymbol{k}^{\boldsymbol{n}}}{\boldsymbol{n}};$$

where, for any $\boldsymbol{\varepsilon} > 0$, the number of functions for which

$$\boldsymbol{M}(\boldsymbol{m}) \le (1 - \boldsymbol{\varepsilon}) \frac{4}{\boldsymbol{k}^4} \frac{\boldsymbol{k}^{\boldsymbol{n}}}{\boldsymbol{n}^2}, \; \boldsymbol{L}(\boldsymbol{m}) \le (1 - \boldsymbol{\varepsilon}) \frac{8}{\boldsymbol{k}^4} \frac{\boldsymbol{k}^{\boldsymbol{n}}}{\boldsymbol{n}},$$

tents to zero as $\boldsymbol{m} \to \infty$ ($\boldsymbol{n} \to \log_{\boldsymbol{k}} \boldsymbol{m}$ for a fixed $\boldsymbol{k}$ or $\boldsymbol{k} \to \sqrt[\boldsymbol{n}]{\boldsymbol{m}}$ for a fixed $\boldsymbol{n}$).

Conclusions

The algebra of generating operations must be chosen with regard for the following.

Hardware implementation. It is reasonable to use the Algebra of Logic and the Multiplicative algebra. In this case the complexity of formal representation can be determined prior to syntheses. Unlike in the Algebra of Logic, in the Multiplicative algebra there is an additional possibility for controlling the values of coefficients, for example, to generate a large number of coefficients with identical values.

Microcontrollers. Additive algebra is suited for programmable devices of medium capacity and without multiplication.

Microprocessors. Finite fields and Integral Rings are useful in this case as its require higher computational resources (both for synthesis and computation).

Polynomial form is preferable, if the number operations are not large.

Nonpolynomial form is preferable with the extended system of operations.